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A COMPLETED NOMENCLATURE FOR THE PRINCIPAL ROULETTES.

F. N. WILLSON.

A COMPLETED NOMENCLATURE

FOR THE

PRINCIPAL ROULETTES

BY

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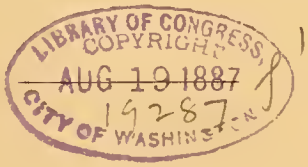

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A COMPLETED NOMENCLATURE FOR THE PRINCIPAL ROULETTES.

That would be an ideal nomenclature in which, from the etymology of the terms chosen, so clear an idea could be obtained of that which is named as to largely anticipate definition, if not, indeed, actually to render it superfluous. This ideal, it need hardly hardly be said, is seldom realized. As a rule we meet with but few self-explanatory terms, and the greater their lack of suggestiveness the greater the need of clear definition. Instances are not wanting of ill-chosen terms and even actual misnomers having become so generally adopted, in spite of an occasional protest, that we can scarcely expect to see them replaced by others more appropriate. Whether this be the case or not, we have a right to expect, especially in the exact sciences, and preëminently in Mathematics, such clearness and comprehensiveness of definition as to make ambiguity impossible. But in this we are frequently disappointed, and notably so in the class of curves we are to consider.

Toward the close of the seventeenth century, the mechanician De la Hire gave the name of Roulette—or roll-traced curve—to the path of a point in the plane of a curve rolling upon any other curve as a base. This suggestive term has been generally adopted, and we may expect its complementary, and equally self-interpreting term, Glissette, to keep it company for all time.

By far the most interesting and important roulettes are those traced by points in the plane of a circle rolling upon another circle in the same plane, such curves having valuable practical applications in mechanism, while their geometrical properties have for centuries furnished an attractive field for investigation to mathematicians.

The Terms *Cycloids* and *Trochoids* are somewhat indiscriminately used as general names for this class of curves. As far as derivation is concerned they are equally appropriate, the former being from κύκλος, circle, and εἶδος,

form; and the latter from τροχός, wheel, and εἶδος. Preference has, however, been given to the term *Trochoids* by several recent writers on mathematics or mechanism, among them Prof. R. H. Thurston and Prof. De Volson Wood; also Prof. A. B. W. Kennedy of England, perhaps best known in this country as the translator of Reuleaux' *Theoretische Kinematik*. Adopting it for the sake of aiding in establishing uniformity in nomenclature I give the following definition:—

If two circles are tangent, either externally or internally, and while one of them remains fixed the other rolls upon it without sliding, the curve described by any point on a radius of the rolling circle, or on a radius produced, will be a Trochoid.

Of these curves the most interesting, both historically and for its mathematical properties, is the *cycloid*, with which all are familiar as the path of a point on the circumference of a circle which rolls upon a straight line, (i.e., the circle of infinite radius).

The term "cycloid," alone, for the locus described, is almost universally employed, although it is occasionally qualified by the adjectives *right* or *common*.

Of almost equally general acceptation, also, although frequently inappropriate,* are the adjectives *curtate* and *prolate*, to indicate trochoidal curves traced by points respectively *without* and *within* the circumference of the rolling circle (or *generator* as it will hereafter be termed) whether it roll upon a circle of finite or infinite radius.

As distinguished from curtate and prolate forms all the other trochoids are frequently called *common*.

Should the fixed circle (called either the *base* or *director*) have an infinite radius, or, in other words, be a straight line, the curtate curve is called by some *the curtate cycloid;* by others, *the curtate trochoid*, and similarly for the prolate forms. Since uniformity is desirable I have adopted the terms which seem to have in their favor the greater number of the authorities consulted, viz., curtate and prolate *trochoid*. It should also be further stated here, with reference to this word "trochoid," that it is usually the termination of the name of every curtate and prolate form of trochoidal curve, the termination *cycloid* indicating that the tracing point is *on* the circumference of the generator.

The general name *Linear Trochoids*, adopted by Prof. Kennedy for the curves having a straight line director, is suggestive and appropriate.

With the base a straight line, the curtate curves consist of a series of loops, while the prolate forms are sinuous, like a wave line; and the same

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* See page 14, first paragraph.

is frequently true when the base is a circle of finite radius. Hence the suggestion of Prof. Clifford that the terms *looped* and *wavy* be employed instead of *curtate* and *prolate*. But we shall see, as we proceed, that they would not be of universal applicability, and that, except with a straight line director, both curtate and prolate curves may be, in form, looped, wavy, or neither.* And we would all agree with Prof. Kennedy that as substitutes for these terms "Prof. Cayley's *kru-nodal* and *ac-nodal* hardly seem adapted for popular use." It is therefore futile to attempt to secure a nomenclature which shall, throughout, suggest both the *form* of the locus and the *mode of its construction*, and we must rest content if we completely attain the latter desideratum.

We have next to consider the trochoids traced during the rolling of a circle upon another circle of finite radius. At this point we find inadequacy in nomenclature, and definitions involving singular anomalies. The earlier definitions have beeen summarized as follows by Prof. R. A. Proctor, in his valuable *Geometry of Cycloids*:—

"The $\left\{\begin{matrix}\text{epicycloid}\\ \text{hypocycloid}\end{matrix}\right\}$ is the curve traced out by a point in the circumference of a circle which rolls without sliding on a fixed circle in the same plane, the two circles being in $\left\{\begin{matrix}\text{external}\\ \text{internal}\end{matrix}\right\}$ contact."

As a specific example of this class of definition I quote the following from a more recent writer:—"If the generating circle rolls on the circumference of a fixed circle, instead of on a fixed line, the curve generated is called an epicycloid if the rolling circle and the fixed circle are tangent externally, a hypocycloid if they are tangent internally." (Byerly, *Differential Calculus*, 1880.)

In accordance with the foregoing definitions every epicycloid is also a hypocycloid, while only some hypocycloids are epicycloids. Salmon, (*Higher Plane Curves*, 1879) makes the following explicit statement on this point:—"The hypocycloid, when the radius of the moving circle is greater than that of the fixed circle, may also be generated as an epicycloid."

To avoid any anomaly, Prof. Proctor has presented the following unambiguous definition:—

"An $\left\{\begin{matrix}\text{epicycloid}\\ \text{hypocycloid}\end{matrix}\right\}$ is the curve traced out by a point on the circumference of a circle which rolls without sliding on a fixed circle in the same plane, the rolling circle touching the $\left\{\begin{matrix}\text{outside}\\ \text{inside}\end{matrix}\right\}$ of the fixed circle."

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* See page 14, first paragraph.

This certainly does away with all confusion between the *epi-* and *hypo-* curves, but we shall find it inadequate to enable us, clearly, to make certain desirable distinctions.

By some writers the term *external epicycloid* is used when the generator and director are tangent externally, and, similarly, *internal epicycloid* when the contact is internal and the larger circle is rolling. Instead of *internal epicycloid* we often find *external hypocycloid* used. It will be sufficient, with regard to it, to quote the following from Prof. Proctor:—"It has hitherto been usual to define it (the hypocycloid) as the curve obtained when either the convexity of the rolling circle touches the concavity of the fixed circle, or the concavity of the rolling circle touches the convexity of the fixed circle. There is a manifest want of symmetry in the resulting classification, seeing that while every epicycloid is thus regarded as an external hypocycloid, no hypocycloid can be regarded as an internal epicycloid. Moreover, an external hypocycloid is in reality an anomaly, for the prefix 'hypo' used in relation to a closed figure like the fixed circle, implies interiorness." To avoid the confusion which it is evident, from the foregoing, has existed, and at the same time to conform to that principle which is always a safe one and never more important than in nomenclature, viz., not to use two words where one will suffice, I prefer reserving the term "epicycloid" for the case of external tangency, and substituting the more recently suggested name *pericycloid* for both "internal epicycloid" and "external hypocycloid." The curtate and prolate forms would then be called *peritrochoids.* By the use of these names and those to be later presented we can easily make distinctions which, without them, would involve undue verbiage in some cases, and, in others, the use of the ambiguous or inappropriate terms to which exception is taken. And the necessity for such distinctions frequently arises, especially in the study of kinematics and machine design. Take, for example, problems like many in the work of Reuleaux already mentioned, relating to the relative motion of higher kinematic pairs of elements, the centroids being circular arcs and the point-paths trochoids. In such cases we are quite as much concerned with the relative position of the rolling and fixed circles as with the form of a point-path. In solving problems in gearing the same need has been felt of simple terms for the trochoidal profiles of the teeth, which should imply the method of their generation.

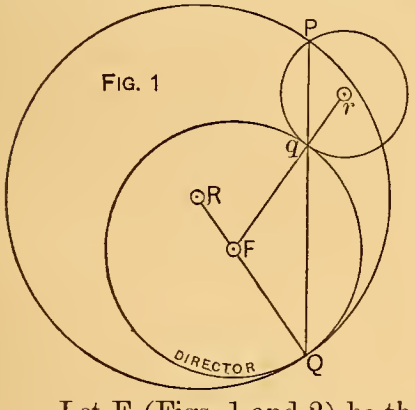
Although they have not, as yet, come into general use, the names pericycloid and peritrochoid appear in the more recent editions of Weisbach and Reuleaux and will undoubtedly eventually meet with universal acceptance.

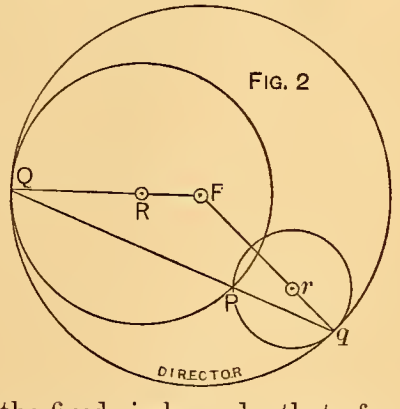
Yet strong objection has been made to the term "pericycloid" by no less an authority than the late eminent mathematician, Prof. W. K. Clifford, who nevertheless adopts the "peritrochoid." I quote the following from his *Elements of Dynamic*:—"Two circles may touch each other so

that each is outside the other, or so that one includes the other. In the former case, if one circle rolls upon the other, the curves traced are called epicycloids and epitrochoids. In the latter case if the inner circle roll on the outer the curves are hypocycloids and hypotrochoids, but if the outer circle roll on the inner, the curves are epicycloids and peritrochoids. We do not want the name pericycloids, because, as will be seen, every pericycloid is also an epicycloid; but there are three distinct kinds of trochoidal curves." As it will later be shown that every peri-*trochoid* can also be generated as an epi-trochoid we can scarcely escape the conclusion that the name *peritrochoid* would also have been rejected by Prof. Clifford, had he been familiar with this property of double generation as belonging to the curtate and prolate forms as well. But it is this very property, possessed also by the *hypo*-trochoids, which necessitates a more extended nomenclature than that heretofore existing, and I am not aware that there has been any attempt to provide the nine terms essential to its completeness. These it is my principal object to present, and that they have not before been suggested I attribute to the fact that the double generation of curtate and prolate trochoidal curves does not seem to have been generally known, being entirely ignored in many treatises which make quite prominent the fact that it is a property of the epi– and hypo-*cycloids*, while, as far as I have seen, the few writers who mention it prove it indirectly, by showing the identity of trochoids with epicyclics and establishing it for the latter.

As it is upon this peculiar and interesting feature that the nomenclature, as now completed, depends, it may be well to give the demonstrations necessary to establish it.

For the epi– and hypo-*cycloid* probably the simplest method of proof is that of Duhamel and which we may call a kinematic, as distinguished from a strictly geometrical, demonstration. It is, in substance, as follows :—





Let F (Figs. 1 and 2) be the centre of the fixed circle, and r that of a rolling circle, the tracing point, P, being *in* the circumference of the latter.

The point of contact, q, is—at the moment that the circles are in the relative position indicated—an instantaneous centre of rotation for every point in the plane of the rolling circle; the line, Pq, joining such point of contact with the tracing point is therefore a normal to the trochoid that the point P is tracing. But if the normal Pq be produced to intersect the fixed circle in a second point, Q, it is evident that the same infinitesimal arc of the trochoid would be described with Q serving as instantaneous centre as when q fulfilled that office. The point, P, will, therefore, evidently trace the same curve, whether it be considered as in the circumference of the circle r, or in that of a second and larger circle, R, tangent to the fixed circle at Q.

It is worth while, in this connection, to note what erroneous ideas with regard to these same loci were held by some writers as late as the middle of this century,—ideas whose falsity it would seem as if the most elementary geometrical construction would have exposed. Reuleaux instances the following statement made by Weissenborn in his *Cyclischen Kurven* (1856): "If the circle described about m_0 roll upon that described about M, and if the describing point, B_0, describe the curve $B_0P_1P_2$ as the inner circle rolls upon the arc B_0b, then, evidently, if the smaller circle be fixed and the larger one rolled upon it in a direction opposite to that of the former rotation, the point of the great circle which at the beginning of the operation coincided with B_0 describes the same line $B_0P_1P_2$." The fallacy of this statement is to us, perhaps, in the light of what has preceded, a little more evident than Weissenborn's deduction; although, as Reuleaux says, "his 'evidently' expresses the usual notion, and the one which is suggested by a hasty pre-judgment of the case. In point of fact B_0 describes the pericycloid $B_0B'B''$, which certainly differs sufficiently from the hypocycloid $B_0P_1P_2$."

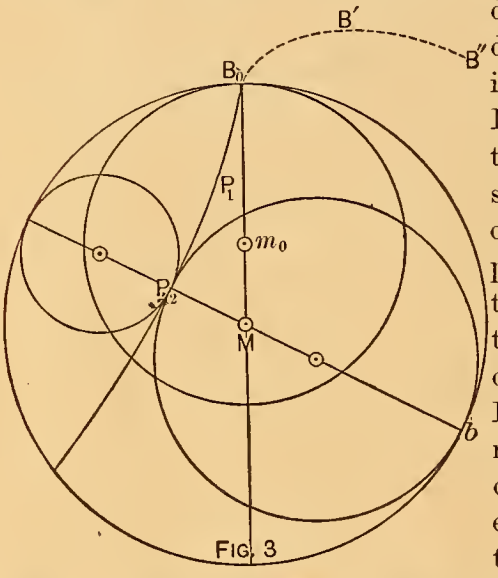


FIG. 3

We have next to consider the curtate and prolate epi-, hypo- and peritrochoids.

As previously stated, I have seen no direct proof that they also possess the same property of double generation, but find the kinematic method equally applicable to them and preferable on account of its simplicity. [It may safely be assumed that all present are so familiar with the ordinary

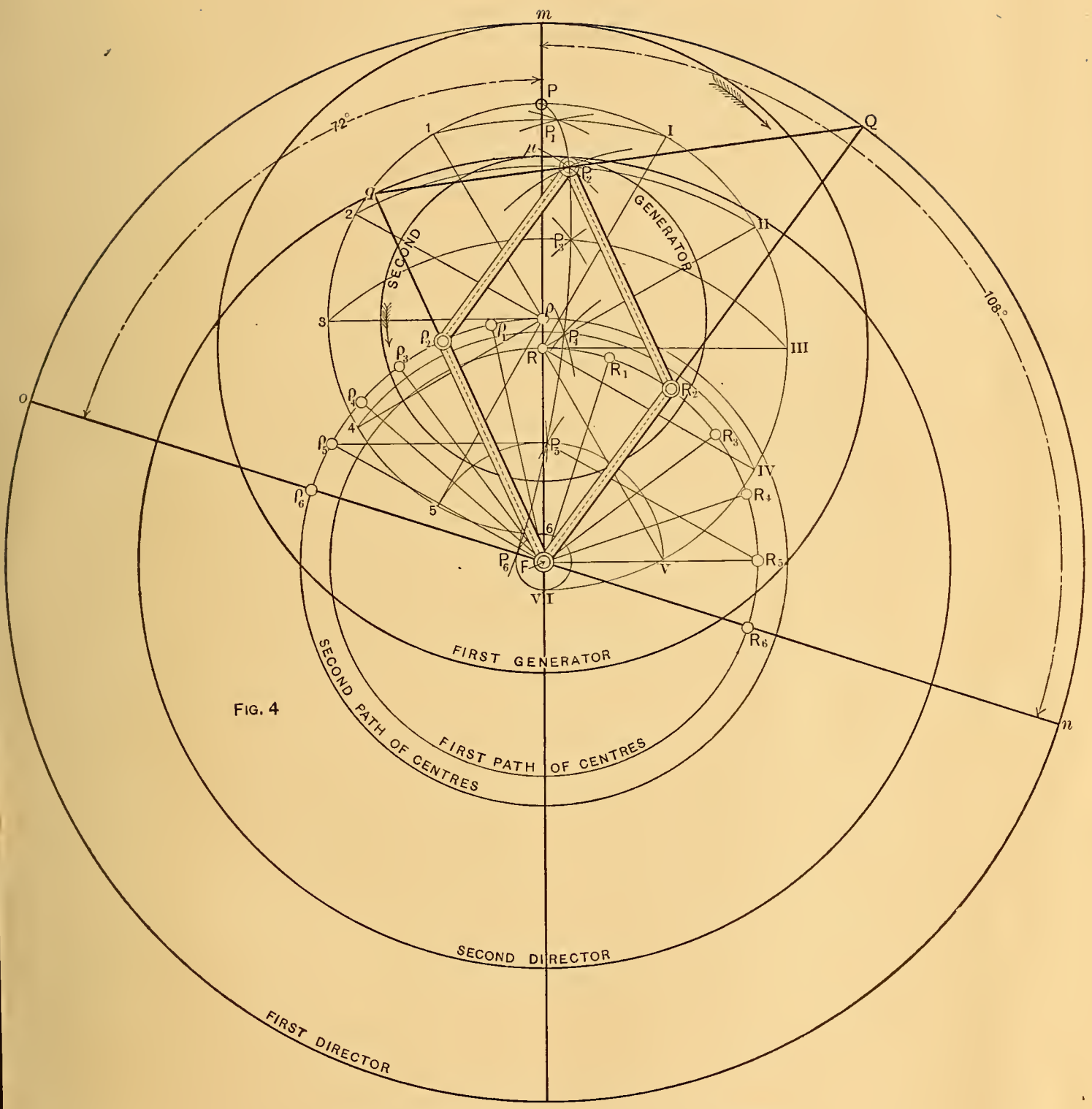


FIG. 4

DOUBLE GENERATION OF HYPOTROCHOIDS.

method of constructing trochoids as to render any explanation of it unnecessary. But as I am expecting to make future use of this paper with some who are but beginners* in the study of these loci, I have given in each figure the geometrical construction of that portion of the curve traced during the turning of each generator through an angle of 180° about its own centre.]

For the hypotrochoids, let R, Fig. 4, be the centre of the first rolling circle or generator, F that of the first director and P the initial position of the tracing point. The initial point of tangency of generator and director is *m*. Let the generator roll over *any* arc of the director, as *m*Q. The centre, R will then be found at R_2 and the tracing point P at P_2. The point of contact, Q, will then be the instantaneous centre of rotation for P_2, and P_2Q will, therefore, be a normal to the trochoid for that particular position of the tracing point.

The motion of P is evidently circular about R, while that of R is in a circle about F. The curve PP_1P_2. . . . . P_6 is that portion of the hypotrochoid which is described while P describes an arc of 180° about R, the latter meanwhile moving through an arc of 108° about F, the ratio of the radii being 3:5.

Now while tracing the curve indicated, the point P can be considered as rigidly connected with a second point, ρ, about which it also describes a circle, ρ meanwhile (like R) describing a circle about F. Such a point may be found as follows. Take any position of P, as P_2, and join it with the corresponding position of R, as R_2; also join R_2 to F. Let us then suppose P_2R and R_2F to be adjacent links of a four-link mechanism. Let the remaining links, $F\rho_2$ and ρ_2P_2, be parallel and equal to P_2R_2 and R_2F respectively. It is then evident that in the turning of R_2 about F, and of P_2 about R_2, the point ρ_2 will describe a circle about F, and that the motion of P_2 with respect to ρ_2 will be in a circular arc of radius ρ_2P_2. We

*Such will need the following explanation of Fig. 4:

It must be evident that, in the rolling of the first generator upon its director, the centre R of the former will describe a circle about F and that the tracing point P will first approach F and then recede from it; also that P will be nearest to F when it has turned 180° about the centre R. This will occur when 180° of the circumference of the generator has come in contact with that of the director. But 108° of arc of the latter is equal in length to 180° of the former, equal arcs on unequal circles being subtended by angles at the centre which are inversely proportional to the radii. Drawing through F a line making an angle of 108° with the line F*m* which joins F with the initial point of contact of the rolling and fixed circles, we find P upon it at P_6, R meanwhile having reached R_6 on the same line.

To find intermediate positions of P divide into the same number of equal parts both the path of centres R-R_6 and the semi-circumference P-VI drawn with radius RP. In the figure six equal divisions have been taken, giving us R_1 R_2. . . R_6 on the former and I, II. . . VI on the latter. With F as a centre draw arcs through the points I, II, III, &c., in succession.

After turning one-sixth of 180° about R, *i. e.* through the angle PRI, the point P will evidently be nearer to F by the difference between FP and FI and somewhere on the arc

may therefore with equal correctness consider ρ_2 as the centre of a generator carrying the point P_2, and $\rho_2 F$ a new line of centres, intersected by the normal P_2Q in a second instantaneous centre, q, which, in strictest analogy with Q, divides the line of centres on which it lies into segments, $\rho_2 q$ and Fq, which are the radii of the second generator and director respectively; q being (like Q) the point of contact of the rolling and fixed circles for the instant that the tracing point is at P_2. The second generator and director, having $\rho_2 q$ and qF respectively for their radii, are represented in their initial positions, ρ being the centre of the former and μ the initial point of contact. The second generator rolls in the opposite direction to the first.

It is important to notice that whereas the tracing point is in the first case *within* the generator and therefore traces the curve as a *prolate* hypotrochoid, it is *without* the second generator and describes the same curve as a *curtate* hypotrochoid. If we now let R and F denote no longer the centres, but the *radii*, of the rolling and fixed circles, respectively, we have for the first generator and director $2R > F$ and for the second $2R < F$.

It has occurred to me that a distinction can very easily be made between trochoids generated under these two opposite relations of radii, by using the simple and suggestive term *major hypotrochoid* when 2R is greater than F, and *minor hypotrochoid* when the opposite relation prevails. We would then say that the preceding demonstration had established the identity of a major prolate with a minor curtate hypotrochoid.

Similarly the identity of major curtate and minor prolate forms could be shown.

If the tracing point were *on* the circumference of the generator the trochoids traced would be, by the new nomenclature, major and minor hypo-*cycloids*.

It is worth noticing that for both hypo-*cycloids* and hypo-*trochoids* the *centre* F is the same for both generations, and that the *radius* F is also constant for both generations of a hypo-*cycloid* but variable for those of a hypo-

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drawn through I, with centre F. It will also be at a distance RP from R, which is now at R_1, having turned through one-sixth of $R—R_6$. P will therefore be at P_1, the intersection of arc through I and an arc of radius PR and centre R_1. Successive positions of P can be similarly found.

If P be considered as carried by the second generator the construction is in every particular analogous to that just given. We find 72° as the value of the angle mFo to be set off at the centre of the second director in order to cut off an arc on its circumference equal to 180° of the new generator, and especial attention is called to the fact that the angle mFo will always be the supplement of the angle mFn. The path of centres $\rho—\rho_6$ is as before, divided into six equal parts, that number of divisions being taken merely for convenience, since we can also divide into six equal parts the semi-circumference drawn with radius ρP by simply continuing the arcs having centre F and passing through I, II, III, &c., until they intersect it at 1, 2, 3, &c. This last is the most interesting feature of the geometrical construction, yet one which, evidently, could not be otherwise if the rolling of two different generators is to give the same curve.

trochoid. Having given the radii of generator and director for the construction of a hypo-trochoid, the method just illustrated will always give the lengths of the radii of the second rolling and fixed circles. The accuracy of the values thus obtained may be checked by simple formulae derived as follows :—

Radii being given for generation as a *major* hypotrochoid, to find corresponding values for the identical *minor* hypotrochoid.

Let F_1 denote the radius FQ [= F*m*] of the first director.
" F_2 " " " Fq [= Fμ] " " second "
" r " " " R_2Q [= R*m*] " " first generator.
" ρ " " " $\rho_2 q$ [= $\rho\mu$] " " second "
" tr " " *tracing radius* of the first generation, i. e., the distance R_2P_2 (or RP) of tracing point from centre of first generator.

Let $t\rho$ then equal the second tracing radius = ρ_2P_2 = ρP. From similar triangles QF*q* and QR_2P_2 we have—

$$F_2 : F_1 :: tr : r$$

whence $F_2 = \frac{F_1\,(tr)}{r}$ ..... (1)

also $\rho = F_2 - tr = \frac{F_1\,(tr)}{r} - tr = tr\left\{\frac{F_1}{r} - 1\right\}$ ... (2)

and $t\rho = \rho_2P_2 = FR_2 = d$, the distance between the centers of first generator and director. ..... (3)

If the radii be given for a *minor* hypotrochoid, we have—

$$FQ : \rho_2P_2 :: Fq : \rho_2 q$$

from which we have, as before,

$$\textit{fixed radius desired} = \frac{\textit{radius of given fixed circle} \times \textit{given tracing radius}}{\textit{radius of given generator}} \quad \ldots (4)$$

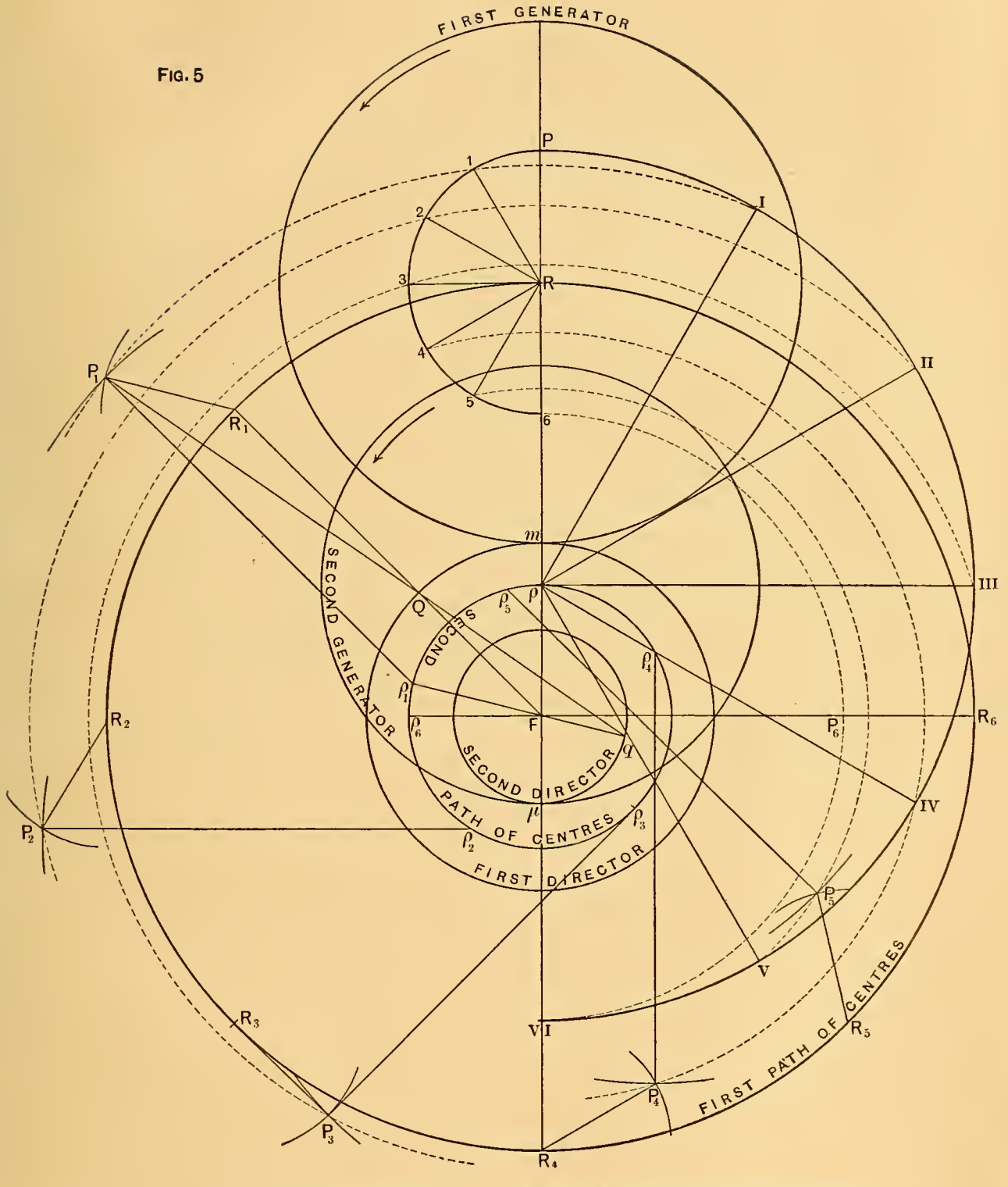
and, similarly, formulae (2) and (3) give the radius of desired generator and the corresponding tracing radius.

With the tracing point *on* the circumference of the generator, if we let R = radius of the latter for a *major* hypocycloid and *r* correspondingly for the *minor* curve, then

for major hypocycloid $R = F - r$ ..... (5)
" minor " $r = F - R$ ..... (6)

For the curves intermediate between the major and minor hypotrochoids, viz., those traced when the diameter of the rolling circle is exactly half that of the fixed circle, a separate division seems essential to completeness, and for such I suggest the general name *medial hypotrochoids.* For these the formulae for double generation are the same as for the "major" and "minor" curves and similarly derived.

Fig. 5



DOUBLE GENERATION OF EPI- AND PERI-TROCHOIDS.

With the tracing point *on* the circumference of the generator these curves reduce to straight lines, diameters of the director. In all other cases they are an interesting exception to what we might naturally expect, being neither looped nor wavy, but *ellipses*. The failure of the terms "looped" and "wavy" to apply to these medial curves is paralleled by that of the adjectives "curtate" and "prolate," since—contrary to the signification of the latter terms—any ellipse generated as a curtate curve is larger than the largest prolate elliptical hypotrochoid having the same director. And as we have seen that, with scarcely an exception, "curtate" and "prolate" apply equally to the same curve, our only reason for retaining them is the fact of their general acceptation as indicative of the location of the tracing point with respect to the circumference of the rolling circle.

Since the medial hypotrochoids are either straight lines or ellipses, we can readily find for them that which we have found it useless to attempt to construct for the other trochoidal curves, viz., simple terms suggestive of their *form*; in fact the names "straight hypocycloid" and "elliptical hypotrochoid" have long been familiar to us all and we have but to incorporate them into the nomenclature we are constructing.

It only remains to show (Fig. 5) that a prolate *epi*-trochoid can be generated as a curtate *peri*-trochoid, and vice versa, for which the demonstration is analogous to that given for the hypo-curves and leads to the following formulae, derived from the similar triangles QFq and QR_1P_1 (the values being supposed to be given for the *epi*-trochoid and desired for the *peri*-trochoid):

$$F_2 = \frac{F_1 (tr)}{r} \quad \ldots\ldots (7)$$

$$\rho = tr \left\{ \frac{F_1}{r} + 1 \right\} \quad \ldots\ldots (8)$$

$t\rho = d =$ distance between centres of given generator and director $= F_1 + r$. $\ldots\ldots (9)$

If given as a *peri*-trochoid and desired as an *epi*-trochoid the tracing radius will again equal the distance between the given centres (in this case, however $= R-F$); the formula for the radius of desired director will be of the same form as equations (1) and (7), but

$$\text{radius of second generator} = tr \left\{ 1 - \frac{F_1}{r} \right\} \quad \ldots\ldots (10)$$

With the tracing point *on* the circumference of the generator, and letting R = radius of the same for a peritrochoid and r for an epitrochoid we have

for the epicycloid, $r = R - F$ $\ldots\ldots (11)$

" the pericycloid, $R = F + r$ $\ldots\ldots (12)$

Although I have by no means discussed all the terms under which trochoidal curves have been previously known, yet enough has been said to indicate how annoying the inconsistencies and ambiguities of former names and definitions have been, and how desirable it is that some adequate and consistent system should be generally adopted. If it has been my good fortune to contribute materially towards reaching such a satisfactory conclusion of the whole matter, if not, indeed, to have settled the question entirely, my object will have been accomplished.

In conclusion I give the complete nomenclature in tabular form, the arrangement having been suggested by that of Kennedy and being both a modification and extension of his ingenious scheme. It does away, to a considerable extent, with the necessity for definition.

The foregoing demonstrations have established the identity of the curves whose names are preceded by the same letter.

TROCHOIDS.

Position of Tracing or Describing Point.	Circle rolling upon straight line.	Circle rolling upon circle.				
		External contact.	Internal contact.			
			Larger circle rolling.	Smaller circle rolling.		
	Linear Trochoids.	Epitrochoids.	Peritrochoids.	2 R > F* Major Hypotrochoids.	2 R < F* Minor Hypotrochoids.	2 R=F* Medial Hypotrochoids.
On circumference of rolling circle.	Cycloid.	(a) Epicycloid.	(a) Pericycloid.	(d) Major Hypocycloid.	(d) Minor Hypocycloid.	Straight Hypocycloid.
Within Circumference.	Prolate Trochoid.	(b) Prolate Epitrochoid.	(c) Prolate Peritrochoid.	(e) Major Prolate Hypotrochoid.	(f) Minor Prolate Hypotrochoid.	(g) Prolate Elliptical Hypotrochoid.
Without Circumference.	Curtate Trochoid.	(c) Curtate Epitrochoid.	(b) Curtate Peritrochoid.	(f) Major Curtate Hypotrochoid.	(e) Minor Curtate Hypotrochoid.	(g) Curtate Elliptical Hypotrochoid.

**R=radius of rolling circle, F of fixed circle.*

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